

On The b-Chromatic Number of Regular Graphs Without 4-Cycle

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Abstract

The b-chromatic number of a graph G , denoted by $\varphi(G)$, is the largest integer k that G admits a proper k -coloring such that each color class has a vertex that is adjacent to at least one vertex in each of the other color classes. We prove that for each d -regular graph G which contains no 4-cycle, $\varphi(G) \geq \lfloor \frac{d+3}{2} \rfloor$ and if G has a triangle, then $\varphi(G) \geq \lfloor \frac{d+4}{2} \rfloor$. Also, if G is a d -regular graph which contains no 4-cycle and $\text{diam}(G) \geq 6$, then $\varphi(G) = d + 1$. Finally, we show that for any d -regular graph G which does not contain 4-cycle and $\kappa(G) \leq \frac{d+1}{2}$, $\varphi(G) = d + 1$.

Keywords: b-chromatic number, girth, diameter, vertex connectivity.

Subject classification: 05C15

1 Introduction

All graphs considered in this paper are finite and simple (undirected, loopless and without multiple edges). Let $G = (V, E)$ be a graph. A coloring (proper coloring) of G is called a b-coloring of G if each color class contains a vertex that is adjacent to at least one vertex in each of the other color classes. The b-chromatic number of G , denoted by $\varphi(G)$, is the largest integer k such that G admits a b-coloring by k colors. The concept of b-coloring of graphs introduced by Irving and Manlove in 1999 in [16] and has received attention recently, for example, see [1 – 26].

Let G be a graph which is colored. Suppose that v is a vertex of G whose color is c . We say that v is a color-dominating vertex or color c realizes on v if v is adjacent to at least one vertex in each of the other color classes.

It is obvious that for each graph G with maximum degree $\Delta(G)$, $\varphi(G) \leq \Delta(G) + 1$. Kratochvil, Tuza and Voigt in [24] proved that every d -regular graph with at least d^4 vertices satisfies $\varphi(G) = d + 1$. Cabello and Jakovac in [7] reduced d^4 to $2d^3 - d^2 + d$. These bounds show that for each natural number d , there are only finite d -regular graphs such that $\varphi(G) \neq d + 1$. El Sahili and Kouider in [13] asked whether it is true that every d -regular graph of girth at least 5 satisfies $\varphi(G) = d + 1$. Blidia, Maffray and Zemir in [5] proved that b-chromatic number of the Petersen graph is 3 and then conjectured that the Petersen graph is the only exception. They proved this conjecture for $d \leq 6$. Kouider in [20] proved that b-chromatic number of any d -regular graph of girth at least 6 is $d + 1$. El Sahili and Kouider in [13] proved that b-chromatic number of any d -regular graph of girth 5 that contains no 6-cycle is $d + 1$. Cabello and Jakovac in [7] proved a celebrated theorem for the b-chromatic

number of regular graphs of girth 5 which guarantees that b-chromatic number of d -regular graphs with girth at least 5 is bounded below by a linear function of d . They proved that a d -regular graph with girth at least 5 has b-chromatic number at least $\lfloor \frac{d+1}{2} \rfloor$. Also, they proved that for except small values of d , every connected d -regular graph that contains no 4-cycle and its diameter is at least d , has b-chromatic number $d + 1$.

In this paper, we discuss about b-chromatic number of d -regular graphs with no 4-cycles. First, in Section 2 we prove that $\varphi(G) \geq \lfloor \frac{d+3}{2} \rfloor$ and if G has a triangle, then $\varphi(G) \geq \lfloor \frac{d+4}{2} \rfloor$. These lower bounds are sharp for the Petersen graph. Also, in Section 3 we prove that if $\text{diam}(G) \geq 6$, then $\varphi(G) = d + 1$. Finally, in Section 3 we show that if $\kappa(G) \leq \frac{d+1}{2}$, then $\varphi(G) = d + 1$. This upper bound for vertex connectivity is sharp for the Petersen graph. Through this paper, for each natural number n , define $[n] := \{i | i \in \mathbb{N}, 1 \leq i \leq n\}$.

2 A Lower Bound

There are many d -regular graphs that contain 4-cycles and their b-chromatic number is not $d+1$. The most famous graphs with this property are complete bipartite graphs $K_{d,d}$ whose b-chromatic number equals 2, independent from d . Cabello and Jakovac in [7] proved that the b-chromatic number of d -regular graphs with girth at least 5 is bounded below by a linear function of d . They proved that a d -regular graph with girth at least 5 has b-chromatic number at least $\lfloor \frac{d+1}{2} \rfloor$. In this regard, they proved the following technical lemma.

Lemma 1. *Let H be a bipartite graph with partitions U and V such that $|U| = |V|$. Let $u^* \in U$ and $v^* \in V$. If for each vertex $x \in V(H) \setminus \{u^*, v^*\}$, $\deg_H(x) \geq \frac{|V|}{2}$, $\deg_H(u^*) > 0$ and $\deg_H(v^*) > 0$, then H has a perfect matching.*

With a slice refinement, we obtain the following lower bounds.

Theorem 1. *Let G be a d -regular graph that contains no 4-cycle. Then $\varphi(G) \geq \lfloor \frac{d+3}{2} \rfloor$. Besides, If G has a triangle, then $\varphi(G) \geq \lfloor \frac{d+4}{2} \rfloor$. These lower bounds are sharp for the Petersen graph.*

Proof. There is nothing to prove when $d \in \{0, 1, 2\}$. So, let us suppose that $d \geq 3$. Let v be an arbitrary vertex of G . Since G contains no 4-cycle, the maximum degree of the induced subgraph of G on $N_G(v)$ is at most one. If d is odd, there exists some vertex of $N_G(v)$ that is not adjacent to any vertex of $N_G(v)$. If d is even, let v_1, v_2, \dots, v_d be an arbitrary ordering of $N_G(v)$ and if d is odd, let v_1, v_2, \dots, v_d be an arbitrary ordering of $N_G(v)$ such that $v_{\frac{d+1}{2}}$ does not have any neighbors in $N_G(v)$.

We color all vertices of $N_G(v) \cup (\bigcup_{j=1}^{\lfloor \frac{d+1}{2} \rfloor} N_G(v_j))$ by the colors of the set $[d+1]$ such that each vertex of $\{v\} \cup \{v_i | 1 \leq i \leq \lfloor \frac{d+1}{2} \rfloor\}$ sees all colors of $[d+1]$ on its closed neighborhood. First, we color the vertex v by color $d+1$ and for each $1 \leq i \leq d$, we color v_i by color i . For each $1 \leq i \leq d$, let $V_i := N_G(v_i) \setminus (\{v\} \cup N_G(v))$, $C_i := [d] \setminus (\text{the set of colors that are appeared on } \{v_i\} \cup (N_G(v_i) \cap N_G(v)))$ and

$S_i := \{v\} \cup N_G(v) \cup (\bigcup_{j=1}^i V_j)$. Obviously, $|V_i| = |C_i| = d - 1$ or $|V_i| = |C_i| = d - 2$. Also, $|V_i| = |C_i| = d - 2$ if and only if $|N_G(v_i) \cap N_G(v)| = 1$. Also, since G contains no 4-cycle, for each $1 \leq i, j \leq d$, if $i \neq j$, then $V_i \cap V_j = \emptyset$. Now, we follow $\lfloor \frac{d+1}{2} \rfloor$ steps inductively. For each $1 \leq i \leq \lfloor \frac{d+1}{2} \rfloor$, at i -th step, we only color all vertices of V_i by all colors of C_i injectively. Suppose by induction that $1 \leq i \leq \lfloor \frac{d+1}{2} \rfloor$ and for each $1 \leq k \leq i - 1$, at k -th step, we have only colored all vertices of V_k by all colors of C_k injectively in such a way that the resulting partial coloring on S_k is a proper partial coloring. Now, at i -th step, we want to color only all vertices of V_i by all colors of C_i injectively in such a way that the resulting partial coloring on S_i be a proper partial coloring. We consider a bipartite graph H_i with one partition V_i and the other partition C_i that a vertex $x \in V_i$ is adjacent to a color $c \in C_i$ in the graph H_i if and only if (in the graph G) x does not have any neighbors in S_{i-1} already colored by c . Such a coloring of all vertices of V_i by all colors of C_i (as mentioned) exists if and only if H_i has a perfect matching.

Let x be an arbitrary element of V_i . The set of neighbors of x in the graph G that were already colored, is a subset of $\{v_i\} \cup (\bigcup_{j=1}^{i-1} V_j)$. Since G contains no 4-cycle, for each $1 \leq j \leq i - 1$, x has at most one neighbor in V_j . Also, the color of the vertex v_i does not belong to C_i . Therefore, $\deg_{H_i}(x) \geq |C_i| - (i - 1)$. Also, since for each $1 \leq j \leq i - 1$, v_j sees all colors of $[d + 1]$ on its closed neighborhood, each color of $[d + 1]$ appears at most once on V_j . Therefore, for each $c \in C_i$, $\deg_{H_i}(c) \geq |V_i| - (i - 1)$. Hence, for each $a \in V(H_i)$, $\deg_{H_i}(a) \geq |V_i| - (i - 1)$. Since $|V_i| \geq d - 2$, if $1 \leq i \leq \frac{d}{2}$, then $|V_i| - (i - 1) \geq \frac{|V_i|}{2}$. Therefore, $\deg_{H_i}(a) \geq \frac{|V_i|}{2}$. If $\frac{d}{2} < i \leq \lfloor \frac{d+1}{2} \rfloor$, then d is odd and $i = \frac{d+1}{2}$. Since $v_{\frac{d+1}{2}}$ does not have any neighbors in $N_G(v)$, $|V_{\frac{d+1}{2}}| = d - 1$ and therefore, $\deg_{H_{\frac{d+1}{2}}}(a) \geq |V_{\frac{d+1}{2}}| - (\frac{d+1}{2} - 1) = \frac{d-1}{2} = \frac{|V_{\frac{d+1}{2}}|}{2}$.

So, Lemma 1 implies that H_i has a perfect matching and we are done. We conclude that there exists a partial coloring on $N_G(v) \cup (\bigcup_{j=1}^{\lfloor \frac{d+1}{2} \rfloor} N_G(v_i))$ by all colors of the set $[d + 1]$ such that each vertex of $\{v\} \cup \{v_i | 1 \leq i \leq \lfloor \frac{d+1}{2} \rfloor\}$ sees all colors of $[d + 1]$ on its closed neighborhood. This coloring extends to a coloring of G greedily. In the current coloring of G , all colors of the set $[\lfloor \frac{d+1}{2} \rfloor] \cup \{d + 1\}$ are realized. If there exists a color $c \in [d + 1]$ that does not realize in this coloring, then for each vertex a in $V(G)$, which is colored by c , there exists a color (like e) in $[d + 1] \setminus \{c\}$ such that a does not have any neighbors in G colored by e . We exchange the color of the vertex a by color e (we recolor the vertex a by color e). The resulting coloring is a coloring of G such that the color of each vertex whose color was different from c in the previous coloring, has not been changed. Also, the color c does not appear in the current coloring and therefore, the number of colors reduces. Each vertex that was a color-dominating vertex in the previous coloring, is again a color-dominating vertex in the current coloring. Repeating this procedure, there exists a b-coloring of G with at least $\lfloor \frac{d+3}{2} \rfloor$ colors and therefore, $\varphi(G) \geq \lfloor \frac{d+3}{2} \rfloor$.

Now, assume that G has a triangle. If d is odd, then $\lfloor \frac{d+3}{2} \rfloor = \lfloor \frac{d+4}{2} \rfloor$. Consequently, we suppose that d is even. Choose a vertex v that is in a triangle. Let v_1, v_2, \dots, v_d be an arbitrary ordering of $N_G(v)$ such that $\{v_1, v_{\frac{d+2}{2}}\} \in E(G)$. There are not any edges between V_1 and $V_{\frac{d+2}{2}}$, otherwise v_1 and $v_{\frac{d+2}{2}}$ will be contained in a

C_4 . Therefore, degree of each vertex of $H_{\frac{d+2}{2}}$ is at least $(d-2) - (\frac{d+2}{2} - 2) = \frac{|V_{\frac{d+2}{2}}|}{2}$, as desired. ■

For each edge $\{u, v\} \in E(G)$, let $\max_{\{u, v\}}$ be the maximum number of 5-cycles such that the intersection of any two different elements of them is the edge $\{u, v\}$. Also, for each path P of length two where $V(P) = \{u, v, w\}$, set \max_P , the maximum number of 5-cycles such that the intersection of any two different elements of them is the path P . The following theorems can be proved similarly. We omit their proofs for the sake of brevity.

Theorem 2. *Let G be a d -regular graph that contains no 4-cycle. If there exists a vertex $v \in V(G)$ such that for each $\{u, v\} \in E(G)$, the number of 5-cycles that contain the edge $\{u, v\}$ is less than or equal to $\frac{d-2}{2}$, then $\varphi(G) = d + 1$. Besides, if girth of G is 5, $\frac{d-2}{2}$ can be replaced by $\frac{d-1}{2}$.*

Theorem 3. *Let G be a d -regular graph that contains no 4-cycle. If there exists a vertex $v \in V(G)$ such that for each $\{u, v\} \in E(G)$, $\max_{\{u, v\}} \leq \frac{d-2}{2}$ and for each path P of length two, $\max_P \leq \frac{d-2}{2}$, then $\varphi(G) = d + 1$. Besides, if girth of G is 5, $\frac{d-2}{2}$ can be replaced by $\frac{d-1}{2}$.*

3 Diameter

Cabello and Jakovac in [7] proved that for $d \geq 10$, every connected d -regular graph that contains no 4-cycle and its diameter is at least d , has b-chromatic number $d + 1$. In this section, we determine the b-chromatic number of d -regular graphs that contain no 4-cycles and have diameter at least 6.

Theorem 4. *Let G be a d -regular graph that contains no 4-cycle. If $\text{diam}(G) \geq 6$, then $\varphi(G) = d + 1$.*

Proof. There is nothing to prove when $d \in \{0, 1, 2\}$. So, suppose that $d \geq 3$. Since $\text{diam}(G) \geq 6$, there are two vertices v and w of distance at least 6. According to the proof of Theorem 1, there exists a partial coloring (using all colors of the set $[d + 1]$) on a subset V of vertices that are at distance at most two from v such that all colors of the set $[\lfloor \frac{d+3}{2} \rfloor]$ are realized. Also, there exists a partial coloring (using all colors of the set $[d + 1]$) on a subset W of vertices that are at distance at most two from w such that all colors of the set $[d + 1] \setminus [\lfloor \frac{d+3}{2} \rfloor]$ do realize. Since v and w have distance at least 6, there are not any edges between V and W . Therefore, these two partial colorings do not conflict and therefore, they form a partial coloring on $V \cup W$ that all colors of the set $[d + 1]$ are realized. This partial coloring extends to a coloring of G greedily and therefore, $\varphi(G) = d + 1$. ■

4 Vertex Connectivity

Vertex connectivity of a graph G , denoted by $\kappa(G)$, is the minimum cardinality of a subset U of $V(G)$ such that $G \setminus U$ is either disconnected or a graph with only one

vertex. It is well-known that for each graph G , $\kappa(G) \leq \delta(G)$ where $\delta(G)$ denotes the minimum degree of G . In this section, we show that if G is a d -regular graph that contains no 4-cycle and $\kappa(G) \leq \frac{d+1}{2}$, then $\varphi(G) = d + 1$. This upper bound for vertex connectivity is sharp in the sense that the vertex connectivity of the Petersen graph is $\frac{d+1}{2} + 1$ although its b-chromatic number is not $d + 1$.

Theorem 5. *Let G be a d -regular graph that contains no 4-cycle. If $\kappa(G) \leq \frac{d+1}{2}$, then $\varphi(G) = d+1$. This upper bound for vertex connectivity is sharp for the Petersen graph.*

Proof. There is nothing to prove when $d \in \{0, 1, 2\}$. Also, Jakovac and Klavzar in [18] showed that the only cubic graph that contains no 4-cycle and its b-chromatic number is not 4 is the Petersen graph. Since the vertex connectivity of the Petersen graph is 3, the proof is complete for $d = 3$. So, suppose that $d \geq 4$. Let U be a set of minimum cardinality such that $G \setminus U$ is either disconnected or a graph with only one vertex. Obviously, $|U| = \kappa(G)$. Since G is not a complete graph, $G \setminus U$ is disconnected. Let G_1 and G_2 be two distinct connected components of $G \setminus U$. Since $\kappa(G) \leq \frac{d+1}{2} < d$, $|G_i| \geq 2$ ($i = 1, 2$). For each $i \in \{1, 2\}$, we prove that there exists some $a_i \in G_i$ such that there are no edges between a_i and U . In this regard, we consider two cases, $d \in \{4, 6, 7, 8, 9, \dots\}$ or $d = 5$.

Case 1) The case $d \geq 4$ and $d \neq 5$: There is nothing to prove when $U = \emptyset$ (or equivalently G is disconnected). So, we consider the case $U \neq \emptyset$. There exists $b_i \in G_i$ such that $|N_G(b_i) \cap U| \leq \frac{\kappa(G)+1}{2}$, otherwise since $|G_i| \geq 2$ and G contains no 4-cycle, inclusion-exclusion principle implies that $|U| > 2(\frac{\kappa(G)+1}{2}) - 1 = \kappa(G)$, a contradiction. Let $U := \{u_j | 1 \leq j \leq \kappa(G)\}$ and for each $1 \leq j \leq \kappa(G)$, set $A_j^i := \{x | x \in G_i, \{x, u_j\} \in E(G)\}$. For each $1 \leq j \leq \kappa(G)$, b_i has at most one neighbor in A_j^i , otherwise b_i and u_j will be contained in a C_4 . Therefore, $|N_G(b_i) \cap (U \cup (\bigcup_{j=1}^{\kappa(G)} A_j^i))| \leq \frac{\kappa(G)+1}{2} + \kappa(G) = \frac{3\kappa(G)+1}{2}$. If $d = 4$, $\kappa(G) \leq \frac{d+1}{2} = \frac{5}{2}$ and therefore, $\kappa(G) \leq 2$. Hence, $|N_G(b_i) \cap (U \cup (\bigcup_{j=1}^{\kappa(G)} A_j^i))| \leq \frac{7}{2} < 4$ and therefore, there exists some vertex in $N_G(b_i)$ which is not in $U \cup (\bigcup_{j=1}^{\kappa(G)} A_j^i)$. Accordingly there exists some $a_i \in G_i$ such that $N_G(a_i) \cap U = \emptyset$. If $k > 5$, then $\frac{3\kappa(G)+1}{2} \leq \frac{\frac{3}{2}(d+1)+1}{2} = d + \frac{5-d}{4} < d$, hence, there exists some $a_i \in G_i$ such that $N_G(a_i) \cap U = \emptyset$.

Case 2) The case $d = 5$: If $U = \emptyset$, there is nothing to prove. If $U \neq \emptyset$, similar to the previous case, there exists $b_i \in G_i$ such that $|N_G(b_i) \cap U| \leq \frac{\kappa(G)+1}{2}$. Also, let $U := \{u_j | 1 \leq j \leq \kappa(G)\}$ and for each $1 \leq j \leq \kappa(G)$, define $A_j^i := \{x | x \in G_i, \{x, u_j\} \in E(G)\}$. Since for each $1 \leq j \leq \kappa(G)$, b_i has at most one neighbor in A_j^i , $|N_G(b_i) \cap (U \cup (\bigcup_{j=1}^{\kappa(G)} A_j^i))| = |(N_G(b_i) \cap U) \cup (N_G(b_i) \cap (\bigcup_{j=1}^{\kappa(G)} A_j^i))| \leq |N_G(b_i) \cap U| + |N_G(b_i) \cap (\bigcup_{j=1}^{\kappa(G)} A_j^i)| \leq \frac{\kappa(G)+1}{2} + \kappa(G)$. Obviously if $\kappa(G) < 3$ or $|N_G(b_i) \cap U| < 2$ or $|N_G(b_i) \cap (\bigcup_{j=1}^{\kappa(G)} A_j^i)| < 3$, then $|N_G(b_i) \cap (U \cup (\bigcup_{j=1}^{\kappa(G)} A_j^i))| < 5$ and we are done. If none of these three conditions hold, we conclude that $\kappa(G) = 3$, $|N_G(b_i) \cap U| \geq 2$, and $|N_G(b_i) \cap (\bigcup_{j=1}^{\kappa(G)} A_j^i)| \geq 3$. Since $|N_G(b_i) \cap (\bigcup_{j=1}^{\kappa(G)} A_j^i)| \geq 3$, G_i has at least three elements b_{i_1}, b_{i_2} and b_{i_3} different from b_i . Since $|U| = \kappa(G) = 3$ and $\binom{\kappa(G)}{2} = \binom{3}{2} = 3$ and G contains no 4-cycle, there exists some $1 \leq l \leq 3$ such

that $|N_G(b_{i_l}) \cap U| \leq 1$. Since b_{i_l} has at most three neighbors in $\bigcup_{j=1}^{\kappa(G)} A_j^i$, therefore, $|N_G(b_{i_l}) \cap (U \cup (\bigcup_{j=1}^{\kappa(G)} A_j^i))| \leq 4$. Hence, there exists some $a_i \in G_i$ such that $N_G(a_i) \cap U = \phi$.

We conclude that there exist some $a_1 \in G_1$ and some $a_2 \in G_2$ such that $N_G(a_1) \cap U = \phi$ and $N_G(a_2) \cap U = \phi$. For each $i \in \{1, 2\}$, a_i has at most $\kappa(G)$ neighbors in $\bigcup_{j=1}^{\kappa(G)} A_j^i$, therefore, $|N_G(a_i) \setminus (\bigcup_{j=1}^{\kappa(G)} A_j^i)| \geq d - \kappa(G) \geq d - \frac{d+1}{2} = \frac{d-1}{2}$.

For coloring G , we consider two cases, d is even or odd.

The case d is even) In this case, $|N_G(a_i) \setminus (\bigcup_{j=1}^{\kappa(G)} A_j^i)| \geq \frac{d-1}{2}$, so, $|N_G(a_i) \setminus (\bigcup_{j=1}^{\kappa(G)} A_j^i)| \geq \frac{d}{2}$. Let $x_1, x_2, \dots, x_{\frac{d}{2}}$ be $\frac{d}{2}$ different elements of $N_G(a_1) \setminus (\bigcup_{j=1}^{\kappa(G)} A_j^1)$. We color a_1 by color 1 and for each $1 \leq j \leq \frac{d}{2}$, we color x_i by color $i + 1$. Then we color all elements of $N_G(a_1) \setminus \{x_1, x_2, \dots, x_{\frac{d}{2}}\}$ by all colors of the set $[d + 1] \setminus [\frac{d}{2} + 1]$ injectively. Therefore, a_1 is a color-dominating vertex with color 1. For each $1 \leq i \leq \frac{d}{2}$, let $V_i := N_G(x_i) \setminus (\{a_1\} \cup N_G(a_1))$, $C_i := \{2, \dots, d + 1\} \setminus$ (the set of colors that were already appeared on $\{x_i\} \cup (N_G(x_i) \cap N_G(a_1))$), and $S_i := \{a_1\} \cup N_G(a_1) \cup (\bigcup_{j=1}^i V_j)$. Obviously, $V_i \subseteq G_1$. Besides, $|V_i| = |C_i| = d - 1$ or $|V_i| = |C_i| = d - 2$. Also, $|V_i| = |C_i| = d - 2$ if and only if $|N_G(x_i) \cap N_G(a_1)| = 1$. Also, for each $1 \leq i, j \leq \frac{d}{2}$, if $i \neq j$, then $V_i \cap V_j = \phi$. Now, we follow $\frac{d}{2}$ steps inductively. For each $1 \leq i \leq \frac{d}{2}$, at i -th step, we only color all vertices of V_i by all colors of C_i injectively to make x_i a color-dominating vertex. Suppose by induction that $1 \leq i \leq \frac{d}{2}$ and for each $1 \leq k \leq i - 1$, at k -th step, we have only colored all vertices of V_k by all colors of C_k injectively in such a way that the resulting partial coloring on S_k is a proper partial coloring. Now, at i -th step, we want to color only all vertices of V_i by all colors of C_i injectively in such a way that the resulting partial coloring on S_i be a proper partial coloring. We consider a bipartite graph H_i with one partition V_i and the other partition C_i such that a vertex $v \in V_i$ is adjacent to a color $c \in C_i$ in the graph H_i if and only if (in the graph G) v does not have any neighbors in S_i already colored by c . Such a coloring of all vertices of V_i by all colors of C_i (as mentioned) exists if and only if H_i has a perfect matching.

Let v be an arbitrary element of V_i . The set of neighbors of v in the graph G that were already colored, is a subset of $\{x_i\} \cup (\bigcup_{j=1}^{i-1} V_j)$. Since G contains no 4-cycle, for each $1 \leq j \leq i - 1$, v has at most one neighbor in V_j . Also, the color of the vertex x_i does not belong to C_i . Therefore, $\deg_{H_i}(v) \geq |C_i| - (i - 1)$. Also, since for each $1 \leq j \leq i - 1$, x_j is a color-dominating vertex, each color of $[d + 1]$ appears at most once on V_j . Therefore, for each $c \in C_i$, $\deg_{H_i}(c) \geq |V_i| - (i - 1)$. Hence, for each $x \in V(H_i)$, $\deg_{H_i}(x) \geq |V_i| - (i - 1)$. Since $|V_i| \geq d - 2$ and $1 \leq i \leq \frac{d}{2}$, $|V_i| - (i - 1) \geq \frac{|V_i|}{2}$ and therefore, $\deg_{H_i}(x) \geq \frac{|V_i|}{2}$. So, Lemma 1 implies that H_i has a perfect matching and we are done.

We conclude that we have a partial coloring on $S_{\frac{d}{2}}$ such that all colors of $[\frac{d}{2} + 1]$ are realized.

Since G_1 and G_2 are two distinct connected components of $G \setminus U$, there are not any edges between G_1 and G_2 in the graph G . Similarly, there exists a partial coloring on a subset of G_2 such that all colors of $[d + 1] \setminus [\frac{d}{2} + 1]$ do realize. Since there are not any edges between G_1 and G_2 in the graph G , these two different partial colorings, form a partial coloring in the graph G such that all colors of $[d + 1]$

are realized. This partial coloring extends to a coloring of G greedily and therefore, $\varphi(G) = d + 1$.

The case d is odd) In this case, the proof is similar to the previous case. For each $i \in \{1, 2\}$, $|N_G(a_i) \setminus (\bigcup_{j=1}^{\kappa(G)} A_j^i)| \geq \frac{d-1}{2}$. There exists a partial coloring on a subset of G_1 such that all colors of $[\frac{d+1}{2}]$ are realized. Also, there exists a partial coloring on a subset of G_2 such that all colors of $[d+1] \setminus [\frac{d+1}{2}]$ are realized. Since there are not any edges between G_1 and G_2 in the graph G , these two different partial colorings, form a partial coloring in the graph G such that all colors of $[d+1]$ are realized. This partial coloring extends to a coloring of G greedily and therefore, $\varphi(G) = d + 1$. ■

The following theorem can be proved similarly. We omit its proof for the sake of brevity.

Theorem 6. *Let $d \in \mathbb{N} \cup \{0\}$ and G be a d -regular graph that contains no 4-cycle. If $\kappa(G) < \frac{2d-1}{3}$, then $\min\{2\lfloor \frac{d+4}{3} \rfloor, d+1\} \leq \varphi(G) \leq d+1$. Besides, if there exists a set $U \subseteq V(G)$ such that $|U| = \kappa(G)$ and $G \setminus U$ has at least three components, then $\varphi(G) = d+1$.*

Acknowledgement: The author wishes to thank Professor Hossein Hajiabolhassan for his useful comments.

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